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SUMMARY: We are intending to describe our results [T4,T6] in a rather self-contained fashion with some remarks. We shall give no proofs of our theorems.

Let $s\geqq 1$, $k\geqq 1$ be fixed integers, $A:=\{a_0,a_1,\cdots,a_s\}$ be a set, and let $\sigma\in\mathrm{Hom}(A^*,A^*)$ be a monoid homomorphism defined by

$$\sigma(a_0):=a_0{}^k a_1,\ \sigma(a_i):=a_{i+1}\ (1\leqq i\leqq s-1),\ \sigma(a_s):=a_0,$$

where A^* denotes the free monoid generated by A. A homomorphism $\tau\in\mathrm{Hom}(A^*,B^*)$ can be extended to $A^*\cup A^\infty$ by defining $\tau(u_1u_2\cdots u_n\cdots):=\tau(u_1)\tau(u_2)\cdots\tau(u_n)\cdots$ $(u_n\in A)$, where A^∞ denotes the set of all infinite words (to the right) over A. We denote by $\omega=\omega_0\omega_1\cdots\omega_n\cdots=\lim_{m\to\infty}\sigma^m(a_0)$ $(\omega_n\in A)$ the fixed point of the σ prefixed by a_0, where σ^m is the m-fold iteration of the σ (σ^0 is the identity map on $A^*\cup A^\infty$), and $\lim_{m\to\infty}\sigma^m(a_0)$ indicates the word $\omega\in A^\infty$ having $\sigma^m(a_0)$ as its prefix for all $m\geqq 0$. We can show that there exists uniquely a number $\alpha=\alpha(s,k)$ such that

$$f(\alpha)=0,\ \alpha>1,\ f(x):=x^{s+1}-kx^s-1\in\mathbb{Z}[x].$$

Throughout the paper, $H(\underline{h})$ denotes the height of $\underline{h}$, i. e.,

$$H(\underline{h}):=\max\{|h_0|,|h_1|,\cdots,|h_s|\},\ \underline{h}:=(h_0,h_1,\cdots,h_s)\in\mathbb{Z}^{s+1}.$$

<u>Theorem 1</u>. Let $k\geqq s\geqq 1$, $2\leqq g\in\mathbb{Z}$, and let θ_i be numbers defined by

$$\theta_i=\theta_i(g;s,k):=\sum_{\omega_n=a_i} g^{-n-1}\ (0\leqq i\leqq s).$$

Then

$$\left|\sum_{i=0}^{s} h_i\theta_i\right| > \kappa_-/H(\underline{h})^{\alpha(\alpha^s-1)/(\alpha-1)}$$

for all $\underline{h}\in\mathbb{Z}^{s+1}$ with $\underline{h}\neq\underline{0}:=(0,0,\cdots,0)$, and

$$\left|\sum_{i=0}^{s} h_i\theta_i\right| < \kappa_+/H(\underline{h})^{\alpha(\alpha^s-1)/(\alpha-1)}$$

for infinitely many $\underline{h}\in\mathbb{Z}^{s+1}$, where κ_-, κ_+ are positive constants independent of $\underline{h}$.

Corollary 1. The s+1 numbers θ_i are linearly indepndent over $\mathbb{Q}$; the s+2 numbers 1, and θ_i are linearly dependent over $\mathbb{Q}$.

For a given $\tau\in\mathrm{Hom}(A^*,B^*)$ $(B:=\{0,1,\cdots,g-1\},\ 2\leqq g\in\mathbb{N})$, we denote by $M(\tau)$ the $(s+1)\times g$ matrix $(|\tau(a_i)|_j)_{0\leq i\leq s,\,0\leq j\leq g-1}$, where $|u|_j$ is the number of ocurrences of $j(\in B)$ appearing in a word $u(\in B^*)$. For a given word $v\in B^\infty$, ${}_g0.v$ indicates the number defined by the g-adic expansion:

$$ {}_g0.v:=\sum_{n=1}^{\infty} v_n g^{-n} \quad (2\leqq g\in\mathbb{Z}),\quad v=v_1v_2\cdots v_n\cdots \quad (v_n\in B). $$

Theorem 2. Let $(s,k)\in\mathbb{N}^2$ with $(s,k)\neq(t,1)$ $(t\geqq 2)$. Let $\tau\in\mathrm{Hom}(A^*,B^*)$ such that the rank of $M(\tau)$ is greater than one. Then ${}_g0.\tau(\omega)$ is a transcendental number for all g with $2\leqq g\in\mathbb{Z}$.

Corollary 2. Let θ_i be as in Theorem 1. Then the numbers θ_i $(0\leqq i\leqq s)$ are transcendental.

It is remarkable that the numbers θ_i have explicit digits in base g:

Example 1. $(s,k)=(2,2)$: $\alpha^3-2\alpha^2-1=0$, $\alpha=2.2055694\cdots$. Put $a_0=a$, $a_1=b$, $a_2=c$. Then $\sigma(a)=aab$, $\sigma(b)=c$, $\sigma(c)=a$, and

$$\omega=aabaabcaabaabcaaabaabcaabaabcaaabaabaabcaabaabcaaabaabcaabaabcaaabaabaabc\cdots,$$

$$\theta_0={}_g0.1101100110110011101100110110011101101100110110011101100110110011101101100\cdots,$$

$$\theta_1={}_g0.0010010001001000010010001001000010010010001001000010010001001000010010010\cdots,$$

$$\theta_2={}_g0.0000001000000100000001000000100000000001000000100000001000000100000000001\cdots.$$

It follows from Theorem 1 that

$$\Big|\sum_{i=0}^{2} h_i\theta_i\Big|\gg 1/H(\underline{h})^{\mu}$$

holds for all $\underline{h}\in\mathbb{Z}^3$ with $\underline{h}\neq\underline{0}$, and

$$\Big|\sum_{i=0}^{2} h_i\theta_i\Big|\ll 1/H(\underline{h})^{\mu}$$

holds for infinitely many $\underline{h}\in\mathbb{Z}^3$, where $\mu=\alpha(\alpha+1)=7.0701059\cdots$.

Theorem 3. Let $(s,k)\in\mathbb{N}^2$, and let g, θ_i, α be as in Theorem 1. Put $\delta:=\max\{1,|\beta|;\ f(\beta)=0,\ (\alpha\neq\beta\in\mathbb{C})\}$. Then

$$\left|\sum_{i=0}^{s} h_i\theta_i\right| > \kappa_-/\left(H(\underline{h})^{\alpha(\alpha^s-1)/(\alpha-1)}\kappa^{(\log H(\underline{h}))^{(\log\delta)/\log\alpha}}\right)$$

for all $\underline{h}\in\mathbb{Z}^{s+1}$ with $\underline{h}\neq\underline{0}$, and

$$\left|\sum_{i=0}^{s} h_i\theta_i\right| < \kappa_+\left(\kappa^{(\log H(\underline{h}))^{(\log\delta)/\log\alpha}}\right)/H(\underline{h})^{\alpha(\alpha^s-1)/(\alpha-1)}$$

for infinitely many $\underline{h}\in\mathbb{Z}^{s+1}$, where $\kappa_+>0$, $\kappa_->0$, and $\kappa>1$ are constants independent of $\underline{h}$.

Remark 1. $1\leqq\delta<\alpha$ holds. The polynomial $f(x)$ is irreducible for all $s\geqq1$, $k\geqq1$. The α is not always a Pisot number, for instance, $\delta>1$ if $(s,k)=(5,1)$. If the α is a Pisot number, then $\delta=1$, so that the estimates in Theorem 3 turn out to be exact ones as in Theorem 1.

Corollary 3. The number $\theta_i(g;s,k)$ is a non-Liouville number for all $0\leqq i\leqq s$, $2\leqq g\leqq\mathbb{Z}$, and $(s,k)\in\mathbb{N}^2$.

Let $s\geqq1$, and $k\geqq2$ be integers, and let $\tilde{\sigma}\in\mathrm{Hom}(\tilde{A}^*,\tilde{A}^*)$ $(\tilde{A}:=\{a_{-1},a_0,\cdots,a_s\})$ be a monoid homomorphism defined by

$$\tilde{\sigma}(a_{-1}):=a_{-1}{}^{k},\quad \tilde{\sigma}(a_0):=a_0a_1a_{-1}{}^{k-2},$$
$$\tilde{\sigma}(a_j):=a_{-1}a_{j+1}a_{-1}{}^{k-2}\ (1\leqq j\leqq s-1),\quad \tilde{\sigma}(a_s):=a_0a_{-1}{}^{k-1}.$$

We denote by $\tilde{\omega}=\tilde{\omega}_0\tilde{\omega}_1\cdots\tilde{\omega}_n\cdots$ $(\tilde{\omega}_n\in\tilde{A})$ the fixed point of the $\tilde{\sigma}$ prefixed by a_0. Then we can show

Theorem 4. Let $\varphi_i=\varphi_i(g;s,k)$ be numbers defined by

$$\varphi_i:=\varphi(g^{-k^i})\quad(0\leqq i\in\mathbb{Z},\ 2\leqq g\in\mathbb{Z}),$$
$$\varphi(z)=\varphi(z;k,s):=\sum_{\tilde{\omega}_n\in\{a_0,a_s\}} z^n.$$

Then

$$\left|\sum_{i=0}^{s} h_i\varphi_i\right| > \kappa_-/H(\underline{h})^{k(k^s-1)/(k-1)}$$

for all $\underline{h}\in\mathbb{Z}^{s+1}$ with $\underline{h}\neq\underline{0}$, and

$$\left| \sum_{i=0}^{s} h_i \varphi_i \right| < \kappa_+ / H(\underline{h})^{k(k^s-1)/(k-1)}$$

for infinitely many $\underline{h} \in \mathbb{Z}^{s+1}$, where κ_+, κ_- are positive constants independent of $\underline{h}$.

Corollary 4. The number $\varphi(g^{-1})$ is a non-Liouville number for all g with $2 \leqq g \in \mathbb{Z}$.

Corollary 5. Let $i \geqq 0$ be an integer. Then, the s+1 numbers $\varphi_i, \varphi_{i+1}, \cdots, \varphi_{i+s}$ are linearly independent over $\mathbb{Q}$; the s+2 numbers 1, and $\varphi_i, \varphi_{i+1}, \cdots, \varphi_{i+s}$ are linearly dependent over $\mathbb{Q}$.

Example 2. $(s,k)=(1,2)$: Put $a_{-1}=c$, $a_0=a$, $a_1=b$. Then, $\tilde{\sigma}(a)=ab$, $\tilde{\sigma}(b)=ac$, $\tilde{\sigma}(c)=c^2$, and the base-g expansion of the numbers φ_i $(i \geqq 0)$ is given by

$\tilde{\omega}$=a b a c a b c c a b a c c c c c a b a$\cdots$,

φ_0=1. 1 1 0 1 1 0 0 1 1 1 0 0 0 0 0 1 1 1$\cdots$,

φ_1=1. 01 01 00 01 01 00 00 01 01 01 00 00 00 00 00 01 01 01$\cdots$,

φ_2=1.0001000100000000100010000000000001000100010000000000000000000000000100010001$\cdots$,

$\cdots$. The base-g expansion of φ_i / g^{2^i} is written by ${}_g0.\tau(\tilde{\omega})$ with $\tau \in \mathrm{Hom}(\tilde{A}^*, B^*)$ defined by $\tau(a)=\tau(b):=0^{2^i-1}1$, $\tau(b):=0^{2^i}$.

We denote by $\underset{\sim}{\omega}=\underset{\sim}{\omega}_0\omega_1\cdots\underset{\sim}{\omega}_n\cdots$ $(\underset{\sim}{\omega}_n \in A)$ the fixed point of $\underset{\sim}{\sigma}$ prefixed by a_0 with $\underset{\sim}{\sigma} \in \mathrm{Hom}(\tilde{A}^*, \tilde{A}^*)$ defined by

$$\underset{\sim}{\sigma}(a_{-1}):=a_{-1}^{k}, \quad \underset{\sim}{\sigma}(a_0):=a_0 a_{-1}^{k-2} a_1,$$

$$\underset{\sim}{\sigma}(a_j):=a_{-1}^{k-1} a_{j+1} \ (1 \leqq j \leqq s-1), \quad \underset{\sim}{\sigma}(a_s):=a_{-1}^{k-1} a_0, \ k \geqq 2.$$

Then, we can show

Theorem 5. Let $\eta_i=\eta_i(g;s,k)$ be numbers defined by

$$\eta_i := \eta(g^{k^i}) \quad (0 \leqq i \leqq s, \ 2 \leqq g \in \mathbb{Z}),$$

$$\eta(z)=\eta(z;s,k) := \sum_{\underset{\sim}{\omega}_n = a_0} z^{-(n+1)/(k-1)}.$$

Then the same assertion with η_i in place of φ_i in Theorem 4 holds.

Corollary 6. The number $\eta(g)$ is a non-Liouville number for all g with $2 \leqq g \in \mathbb{Z}$.

Corollary 7. Let $i \geqq 0$ be an integer. Then, the s+1 numbers $\eta_i, \eta_{i+1}, \cdots, \eta_{i+s}$ are linearly independent over $\mathbb{Q}$; the s+2 numbers 1, and $\eta_i, \eta_{i+1}, \cdots, \eta_{i+s}$ are linearly dependent over $\mathbb{Q}$.

We refer to the fact that our results have some connection with higher dimensional continued fractions, and certain Boolean equations, cf. [T4, T6]. For instance, let $\chi = \chi(\tilde{\omega}; a_0, a_s)$ with $\tilde{\omega}$ of Theorem 4, where $\chi(u; p, q, r, \cdots)$ denotes the set defined by

$$\chi(u; p, q, r, \cdots) := \{n \geqq 0;\ u_n \in \{p, q, r, \cdots\}\}$$

for $u = u_0 u_1 u_2 \cdots \in A^* \cup A^\infty$ $(u_n \in A)$, $p, q, r, \cdots \in A$. Then, considering an automaton, we can show that $X = \chi$ is a solution of a Boolean equation

(1) $$X = kX \cup (k^{s+1}X + (k^s-1)/(k-1)),\quad \phi \neq X \subset \mathbb{N} \cup \{0\},$$

which implies

(2) $$\varphi(z) = \varphi(z^k) + z^{(k^s-1)/(k-1)} \varphi(z^{k^{s+1}}),$$

where $pX+q := \{px+q;\ x \in X\}$, and $\varphi(z)$ is the function given in Theorem 4. Using the functional eqution (2), we can construct a formal s-dimensional continued fraction:

(3) $$\underline{\phi}(z) = \underline{1} + \cfrac{z}{\underline{1} + \cfrac{z^k}{\underline{1} + \cfrac{z^{k^2}}{\underline{1} + \cdots + \cfrac{z^{k^n}}{\underline{1} + \cdots}}}},$$

where

$$\underline{\phi}(z) := (\phi_1(z), \phi_2(z), \cdots, \phi_s(z)),$$

$$\psi_i(z) =: z^{(k^i-1)/(k-1)} / (\psi_s(z^k)\psi_s(z^{k^2})\cdots\psi_s(z^{k^i})) \quad (1 \leqq i \leqq s-1,\ s \geqq 2),$$

$$\psi_s(z) := \varphi(z)/\varphi(z^k) \quad (s \geqq 1),$$

$$\underline{1} := (0, 0, \cdots, 0, 1) \in \mathbb{R}^{s+1},$$

and

$$\frac{c}{(d_1, d_2, \cdots, d_s)} := c(1/d_s,\ d_1/d_s, \cdots, d_{s-1}/d_s)$$

for given $c,\ d_1,\ d_2,\ \cdots,\ d_s \in \mathbb{C}$ $(d_s \neq 0)$. We can show that the nth convergent of the continued fraction (3) converges component-wise to $\underline{\psi}(z)$ if $z=1/g$ with $2 \leqq g \in \mathbb{Z}$. Using (3), we can give the following expansion of $\underline{\psi}(1/g)$ by the Jacobi-Perron algorithm:

$$\underline{\psi}(1/g) = b_0\underline{1} + \cfrac{1}{b_1\underline{1} + \cfrac{1}{b_2\underline{1} + \cdots}}, \tag{4}$$

$$b_n := g^{k^{n-1}} \quad (n \not\equiv 0 \pmod{s+1}),$$

$$b_n := g^{(k^s-1)(k^n-1)/(k^{s+1}-1)} \quad (n \equiv 0 \pmod{s+1}),$$

cf. [T6]. The continued fraction (4) is a regular one in the sense of Korobov [K], p. 84. We can apply a result due to Korobov ([K], p. 91) on higher dimensional regular continued fractions, and we obtain the exact measure, except for O-constants, of linear independence for the values φ_i as in Theorem 4.

Theorem 5 has a conection with the following continued fraction:

$$\underline{\zeta}(z) = z\underline{1} + \cfrac{1}{z^k\underline{1} + \cfrac{1}{z^{k^2}\underline{1} + \cdots + \cfrac{1}{z^{k^n}\underline{1} + \cdots}}}, \tag{5}$$

where

$$\underline{\zeta}(z)=(\zeta_1(z),\zeta_2(z),\cdots,\zeta_s(z)),$$

$$\zeta_i(z):=1/(\zeta_s(z^k)\zeta_s(z^{k^2})\cdots\zeta_s(z^{k^i}))\quad(1\leqq i\leqq s-1,\ s\geqq 2),$$

$$\zeta_s(z):=\eta(z)/\eta(z^k)\quad(s\geqq 1).$$

We can show that (5) follows from the functional equation

$$\eta(z)=z\eta(z^k)+\eta(z^{k^{s+1}}),\quad \eta(z)=\sum_{m\in X} z^{-m},$$

where X is a solution of the Boolean equation

$$X=(kX-1)\cup k^{s+1}X \tag{6}$$

with countable set $X\subset\mathbb{R}$ such that X is bouded from the left, or from the right. We note that (6) with $k\geqq 3$ has no solutions under $\phi\neq X\subset\mathbb{N}\cup B$ (B is a finite subset of $\mathbb{Z}$), but (6) has a unique solution under $\phi\neq X\subset\mathbb{Z}$, so that X is unbounded from both sides, and $\zeta(z)$ can not be well-defined in the case $\phi\neq X\subset\mathbb{Z}$. Nevertheless, we can show that the equation (6) has a unique solution given by

$$X=(k-1)^{-1}\chi(\underset{\sim}{\omega};a_0)+(k-1)^{-1}$$

with $\underset{\sim}{\omega}$ in Theorem 5 under the condition

$$\phi\neq X\subset(k-1)^{-1}\mathbb{N}.$$

The solution $X=\chi(\tilde{\omega};a_0,a_s)$ of (1) is also a unique one. In general, it will be difficult to solve such a uniqueness problem. In fact, we can show the following

<u>Remark 2</u>. The Boolean equation

$$3X+1=(6X+1)\cup(X\cap(6\mathbb{N}-2))\quad(1\in X\subset\mathbb{N})$$

has a solution $X=\mathbb{N}$. Suppose that $X=\mathbb{N}$ is the unique solution, then one can prove that Syracuse conjecture holds, and vice versa. (Syracuse conjecture is a well-known conjecture as the so called 3x+1 problem, or Collatz problem, or Kakutani's problem, that is still open. The conjecture states that for any given positive integer m, there exist a positive integer n=n(m) satisfing $F^n(m)=1$, where $F(m):=3m+1$ (m: odd), $:=m/2$ (m: even), and F^n denotes the n-fold iteration of the map F.)

It is convenient to use a locally catinative formula among the words $\sigma^n(a_0)$ $(n\geqq 0)$ to construct a higher dimensional continued fraction connected with Theorems 1, 3, cf. [T4]. Let θ_i be as in Theorem 1. Then, the following continued fraction is a corresponding one to Theorems 1, and 3:

$$(7)\qquad \underline{\gamma}=\underline{\gamma}(g;s,k) = \cfrac{1}{c_1\underline{1} + \cfrac{1}{c_2\underline{1} + \cfrac{1}{c_3\underline{1} + \cdots}}} \qquad (g\geqq 2,\ s\geqq 1,\ k\geqq 1)$$

with

$$c_n=c_n(g;s,k)=:g^{f_n}\sum_{h=0}^{k-1} g^{hf_{n+s}},$$

$$f_n:=kf_{n-1}+f_{n-s-1}\ (n\geqq s+2),\ f_n:=(k^n-1)/(k-1)\ (1\leqq n\leqq s+1),$$

and

$$\underline{\gamma}=(\gamma^{(1)},\gamma^{(2)},\cdots,\gamma^{(s)}),$$

$$\gamma^{(i)}:=(\theta_{i-1}+\sum_{j=i}^{s} p_j^{(i)}\theta_i)/\theta_s \quad (1\leqq i\leqq s),$$

$$(p_j^{(i)})_{0\leq i\leq s,\,0\leq j\leq s}:=Q^{-1},\ Q=(q_j^{(i)})_{0\leq i\leq s,\,0\leq j\leq s},$$

$$q_j^{(i)}:=\sum_{h=0}^{k-1} g^{hf_j}\times\sum_{m\in\chi(^R(\sigma^j(a_0));a_i)} g^m \quad (0\leqq i<j\leqq s),$$

$$q_i^{(i)}:= 1\ (0\leqq i\leqq s),\quad q_j^{(i)}:=0\ (0\leqq j<i\leqq s),$$

where we denote by σ the morphism defined in the first paragraph, and by ^Ru the word $u_t u_{t-1}\cdots u_1\in A^*$ for a given word $u_1u_2\cdots u_t$ $(u_m\in A)$. We note that $p_j^{(i)}$ are integers, since $Q\in SL_\pm(s+1;\mathbb{Z})$. Thus, using the result of Korobov [K], we can find the exact mesure of linear independence for the s+1 numbers 1, and $\gamma_1,\gamma_2,\cdots,\gamma_s$ when α is a Pisot number, from where we get Theorem 1, see Remark 1. The continued fraction (7) can be regarded as a higher dimensional version of some of the classical results, see, for example [B], [D], [M], [Bu], see also [K-S-T], [T1-T3]. Related to our transcendence result (Theorem 2), we note that functions $\theta_i(z)$ $(0\leqq i\leqq s)$, $\varphi(z)$, and $\eta(z)$ appearing in our theorems are transcendental functions, which follows from a Theorem due to Fatou [F]. We can prove Theorem 2 by Roth's theorem, estimating the irrationality measure of the number ${}_g0.\tau(\omega)$ from below. The estimate from above is not easy in general. But, in some cases, we can find the exact value of the irrationality measure of the number ${}_g0.\tau(\omega)$ under a minor condition on τ, cf. [T7]. We gave Theorems 1-3, and Theorems 4-5, respectively, in [T4], and in [T6]. We gave a higher dimensional version of the Ramanujan's continued fraction, and the linear independece measure, which is also an exact one except for O-constants, of values of certain q-series, cf. [T5].

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